

# Colorful monochromatic connectivity of random graphs\*

Ran Gu, Xueliang Li, Zhongmei Qin

Center for Combinatorics and LPMC-TJKLC

Nankai University, Tianjin 300071, P.R. China

Email: guran323@163.com, lxl@nankai.edu.cn, qinzhongmei90@163.com

## Abstract

An edge-coloring of a connected graph  $G$  is called a *monochromatic connection coloring* (MC-coloring, for short), introduced by Caro and Yuster, if there is a monochromatic path joining any two vertices of the graph  $G$ . Let  $mc(G)$  denote the maximum number of colors used in an MC-coloring of a graph  $G$ . Note that an MC-coloring does not exist if  $G$  is not connected, and in this case we simply let  $mc(G) = 0$ . We use  $G(n, p)$  to denote the Erdős-Rényi random graph model, in which each of the  $\binom{n}{2}$  pairs of vertices appears as an edge with probability  $p$  independently from other pairs. For any function  $f(n)$  satisfying  $1 \leq f(n) < \frac{1}{2}n(n-1)$ , we show that if  $\ell n \log n \leq f(n) < \frac{1}{2}n(n-1)$  where  $\ell \in \mathbb{R}^+$ , then  $p = \frac{f(n) + n \log \log n}{n^2}$  is a sharp threshold function for the property  $mc(G(n, p)) \geq f(n)$ ; if  $f(n) = o(n \log n)$ , then  $p = \frac{\log n}{n}$  is a sharp threshold function for the property  $mc(G(n, p)) \geq f(n)$ .

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## 1 Introduction

All graphs in this paper are undirected, finite and simple. We follow [2] for graph theoretical notation and terminology not defined here. Let  $G$  be a nontrivial connected graph with an *edge-coloring*  $c : E(G) \rightarrow \{1, 2, \dots, t\}$ ,  $t \in \mathbb{N}$ , where adjacent

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edges may have the same color. A path of  $G$  is said to be a *rainbow path* if no two edges on the path have the same color. A connected graph is *rainbow connected* if there is a rainbow path connecting any two vertices. An edge-coloring of a connected graph is called a *rainbow connection coloring* if it makes the graph rainbow connected. The concept of rainbow connection of graphs was introduced by Chartrand et al. in [5]. The rainbow connection number of a connected graph  $G$ , is the smallest number of colors that are needed in order to make  $G$  rainbow connected. Recently, the rainbow connection colorings have been well-studied, and for details we refer to [10, 11].

In 2011, Caro and Yuster [6] introduced a natural counterpart question of rainbow connection colorings, which is called the monochromatic connection coloring. An edge-coloring of a connected graph  $G$  is called a *monochromatic connection coloring* (MC-coloring, for short) if there is a monochromatic path joining any two vertices. Let  $mc(G)$  denote the maximum number of colors used in an MC-coloring of a graph  $G$ , which called the *monochromatic connection number* of  $G$ . Note that an MC-coloring does not exist if  $G$  is not connected, and in this case we simply let  $mc(G) = 0$ . Denote by  $n$  and  $m$  the number of vertices and edges of graph  $G$ , respectively. Note that by simply coloring the edges of a spanning tree of  $G$  with one color, and assigning the remaining edges other distinct colors, we obtain an MC-coloring of  $G$ , and this MC-coloring provides a straightforward lower bound for  $mc(G)$ , which is summarized that as a theorem below.

**Theorem 1.1** *For any connected graph  $G$ ,  $mc(G) \geq m - n + 2$ .*

In particular,  $mc(G) = m - n + 2$  whenever  $G$  is a tree. Caro and Yuster [6] also showed that there are dense graphs that still meet this lower bound.

**Theorem 1.2** [6] *Let  $G$  be a connected graph with  $n > 3$ . If  $G$  satisfies any of the following properties, then  $mc(G) = m - n + 2$ .*

- (a)  $\overline{G}$  (the complement of  $G$ ) is 4-connected.
- (b)  $G$  is triangle-free.
- (c)  $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$ . In particular, this holds if  $\Delta(G) \leq (n+1)/2$ , and also holds if  $\Delta(G) \leq n - 2m/n$ .
- (d) The diameter of  $G$  is at least 3.
- (e)  $G$  has a cut vertex.

For the upper bounds of  $mc(G)$ , Caro and Yuster [6] gave the following result:

**Theorem 1.3** [6] *Let  $G$  be a connected graph. Then*

- (a)  $mc(G) \leq m - n + \chi(G)$ , where  $\chi(G)$  is the vertex chromatic number of  $G$ .  
(b) if  $G$  is not  $r$ -connected, then  $mc(G) \leq m - n + r$ .

In this paper, we study the number  $mc(G)$  for random graphs. The most frequently occurring probability model of random graphs is the *Erdős-Rényi random graph model*  $G(n, p)$  [7]. The model  $G(n, p)$  consists of all graphs with  $n$  vertices in which the edges are chosen independently and with probability  $p$ . We say an event  $\mathcal{A}$  happens *with high probability* if the probability that it happens approaches 1 as  $n \rightarrow \infty$ , i.e.,  $Pr[\mathcal{A}] = 1 - o_n(1)$ . Sometimes, we say *w.h.p.* for short. We will always assume that  $n$  is the variable that tends to infinity.

Let  $G, H$  be two graphs on  $n$  vertices. A property  $P$  is said to be *monotone* if whenever  $G \subseteq H$  and  $G$  satisfies  $P$ , then  $H$  also satisfies  $P$ . For a graph property  $P$ , a function  $p(n)$  is called a *threshold function* of  $P$  if:

- for every  $r(n) = \omega(p(n))$ ,  $G(n, r(n))$  w.h.p. satisfies  $P$ ; and
- for every  $r'(n) = o(p(n))$ ,  $G(n, r'(n))$  w.h.p. does not satisfy  $P$ .

Furthermore,  $p(n)$  is called a *sharp threshold function* of  $P$  if there exist two positive constants  $c$  and  $C$  such that:

- for every  $r(n) \geq C \cdot p(n)$ ,  $G(n, r(n))$  w.h.p. satisfies  $P$ ; and
- for every  $r'(n) \leq c \cdot p(n)$ ,  $G(n, r'(n))$  w.h.p. does not satisfy  $P$ .

In the extensive study of the properties of random graphs, many researchers observed that there are sharp threshold functions for various natural graph properties. It is well-known that all monotone graph properties have sharp threshold functions; see [3] and [8]. For the property  $rc(G(n, p)) \leq 2$ , Caro et al. [4] proved that  $p = \sqrt{\log n/n}$  is the sharp threshold function. He and Liang [9] studied further the rainbow connectivity of random graphs. Specifically, they obtained that  $(\log n)^{(1/d)}/n^{(d-1)/d}$  is the sharp threshold function for the property  $rc(G(n, p)) \leq d$ , where  $d$  is a constant.

For the monochromatic connectivity of a graph, one aims to find as many colors as possible to keep the graph monochromatically connected. Also, it is natural to ask what kind of graphs have large  $mc(G)$ . That is, we can use a great many colors to make the graph monochromatically connected. Furthermore, what will happen if we require the number of colors to relate with the order of the graph? So it is interesting to consider the threshold function of the property  $mc(G(n, p)) \geq f(n)$ , where  $f(n)$  is a function of  $n$ . For any graph  $G$  with  $n$  vertices and any function  $f(n)$ , having

$mc(G) \geq f(n)$  is a monotone graph property (adding edges does not destroy this property), so it has a sharp threshold function. Realize that for the sharp threshold function for the rainbow connectivity of random graphs, the known results all require that the number of colors is independent of the order of the random graph, but our result does not have that restriction. Our main result is as follows.

**Theorem 1.4** *Let  $f(n)$  be a function satisfying  $1 \leq f(n) < \frac{1}{2}n(n-1)$ . Then*

$$p = \begin{cases} \frac{f(n) + n \log \log n}{n^2} & \text{if } \ell n \log n \leq f(n) < \frac{1}{2}n(n-1), \text{ where } \ell \in \mathbb{R}^+, \\ \frac{\log n}{n} & \text{if } f(n) = o(n \log n). \end{cases}$$

*is a sharp threshold function for the property  $mc(G(n, p)) \geq f(n)$ .*

**Remark.** Note that  $mc(G(n, p)) \leq \frac{1}{2}n(n-1)$  for any probability function  $0 \leq p \leq 1$ , and  $mc(G(n, p)) = \frac{1}{2}n(n-1)$  if and only if  $G(n, p)$  is isomorphic to the complete graph  $K_n$ . Hence we only concentrate on the case  $f(n) < \frac{1}{2}n(n-1)$ .

## 2 Proof of Theorem 1.4

In [6], Caro and Yuster gave the following upper bound for  $mc(G)$ .

**Theorem 2.1** *If the minimum degree of  $G$  is  $\delta(G) = s$ , then  $mc(G) \leq |E(G)| - |V(G)| + s + 1$ .*

In this paper, we use the following version of Chernoff bound:

**Lemma 2.1** [1] (**Chernoff Bound**) *If  $X$  is a binomial random variable with expectation  $\mu$ , and  $0 < \delta < 1$ , then*

$$\Pr[X < (1 - \delta)\mu] \leq \exp\left(-\frac{\delta^2\mu}{2}\right)$$

*and if  $\delta > 0$ , then*

$$\Pr[X > (1 + \delta)\mu] \leq \exp\left(-\frac{\delta^2\mu}{2 + \delta}\right).$$

Throughout the paper “log” denotes the natural logarithm. The following theorem is a classical result on the connectedness of a random graph.

**Theorem 2.2** [7] *Let  $p = (\log n + a)/n$ . Then*

$$\Pr[G(n, p) \text{ is connected}] \rightarrow \begin{cases} e^{-e^{-a}} & \text{if } |a| = O(1), \\ 0 & a \rightarrow -\infty, \\ 1 & a \rightarrow +\infty. \end{cases}$$

From Theorem 2.2 and the definition of sharp threshold functions, we can derive the following corollary immediately.

**Corollary 2.1**  $p = \frac{\log n}{n}$  is a sharp threshold function for  $G(n, p)$  to be connected.

Now we prove Theorem 1.4. According to the range of  $f(n)$ , we have the following two cases.

**Case 1.**  $\ell n \log n \leq f(n) < \frac{1}{2}n(n-1)$ , where  $\ell \in \mathbb{R}^+$ .

To establish a sharp threshold function for a graph property, the proof should be two-folds. We first show one direction.

**Theorem 2.3** *There exists a constant  $C$  such that  $mc\left(G\left(n, C\frac{f(n)+n\log\log n}{n^2}\right)\right) \geq f(n)$  w.h.p. holds.*

*Proof.* Let

$$C = \begin{cases} 5 & \text{if } \ell \geq 1 \\ \frac{5}{\ell} & \text{if } 0 < \ell < 1 \end{cases}$$

and  $p = \frac{f(n)+n\log\log n}{n^2}$ . By Theorem 2.2, it is easy to check that  $G(n, Cp)$  is w.h.p. connected. Let  $\mu_1$  be the expectation of the number of edges in  $G(n, Cp)$ . So

$$\mu_1 = \frac{n(n-1)}{2} \cdot Cp = \frac{C}{2} \left( \frac{n-1}{n} f(n) + (n-1) \log\log n \right).$$

From Lemma 2.1, we have

$$\Pr[|E(G(n, Cp))| < \frac{\mu_1}{2}] \leq \exp\left(-\frac{1}{2} \cdot \frac{1}{4} \mu_1\right) = \exp\left(-\frac{1}{8} \mu_1\right) = o(1).$$

Note that if  $|E(G(n, Cp))| \geq \frac{\mu_1}{2}$ , then by Theorem 1.1, we have that

$$\begin{aligned} mc(G(n, Cp)) &\geq |E(G(n, Cp))| - n + 2 \\ &\geq \frac{\mu_1}{2} - n + 2 \\ &= \frac{C}{4} \left( \frac{n-1}{n} f(n) + (n-1) \log\log n \right) - n + 2 \\ &\geq \frac{5}{4} \left( \frac{n-1}{n} f(n) + (n-1) \log\log n \right) - n + 2 \\ &\geq f(n), \end{aligned}$$

for  $n$  sufficiently large. Thus, we obtain that with probability at least  $1 - \exp(-\frac{1}{8}\mu_1) = 1 - o(1)$ ,  $mc(G(n, Cp)) \geq f(n)$  holds.  $\square$

Next we show the other direction.

**Theorem 2.4**  $mc\left(G\left(n, \frac{f(n)+n \log \log n}{n^2}\right)\right) < f(n)$  w.h.p. holds.

*Proof.* Let  $p = \frac{f(n)+n \log \log n}{n^2}$  and  $\mu_2$  be the expectation of the number of edges in  $G(n, p)$ . We have

$$\mu_2 = \frac{n(n-1)}{2} \cdot p = \frac{1}{2} \left( \frac{n-1}{n} f(n) + (n-1) \log \log n \right).$$

We obtain that

$$\Pr[|E(G(n, p))| > \frac{3}{2}\mu_2] \leq \exp\left(-\frac{\frac{1}{4}\mu_2}{2 + \frac{1}{2}}\right) = \exp\left(-\frac{1}{10}\mu_2\right) = o(1)$$

by Lemma 2.1. If  $G(n, p)$  is not connected, then  $mc(G(n, p)) = 0 < f(n)$ . If  $G(n, p)$  is connected, let  $d$  denote the minimum degree of  $G(n, p)$ , it is obvious that  $d < n$ . If  $|E(G(n, p))| \leq \frac{3}{2}\mu_2$ , then from Theorem 2.1, we have that

$$\begin{aligned} mc(G(n, p)) &\leq |E(G(n, p))| - n + d + 1 \\ &\leq \frac{3}{2}\mu_2 - n + d + 1 \\ &= \frac{3}{4} \left( \frac{n-1}{n} f(n) + (n-1) \log \log n \right) - n + d + 1 \\ &< \frac{3}{4} \left( \frac{n-1}{n} f(n) + (n-1) \log \log n \right) - n + n + 1 \\ &< f(n). \end{aligned}$$

Hence, we have that with probability at least  $1 - \exp(-\frac{1}{10}\mu_2) = 1 - o(1)$ ,  $mc(G(n, p)) < f(n)$  holds.  $\square$

**Case 2.**  $f(n) = o(n \log n)$  or  $f(n)$  is a constant.

By Corollary 2.1 we have that there exist two positive constants  $c_1$  and  $c_2$  such that: for every  $r(n) \geq c_1 \cdot p$ ,  $G(n, r(n))$  is w.h.p. connected; and for every  $r'(n) \leq c_2 \cdot p$ ,  $G(n, r'(n))$  is w.h.p. not connected. Moreover, for  $r(n) \geq c_1 \cdot p$ ,  $|E(G(n, r(n)))| = O(n \log n)$  by Lemma 2.1. Hence,  $mc(G(n, r(n))) \geq |E(G(n, r(n)))| - n + 2 \geq f(n)$ . On the other hand, since  $G(n, r'(n))$  is w.h.p. not connected, for every  $r'(n) \leq c_2 \cdot p$ ,  $mc(G(n, r'(n))) = 0 < f(n)$  w.h.p. holds.

Combining Case 1 and Case 2, our main result follows.  $\blacksquare$

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